Work-fluctuation theorems for a particle in an electromagnetic field

2010 J. Phys. A: Math. Theor. 43 255001

(http://iopscience.iop.org/1751-8121/43/25/255001)

IOP PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND THEORETICAL

J. Phys. A: Math. Theor. **43** (2010) 255001 (17pp) doi:10.1088/1751-8113/43/25/255001

Work-fluctuation theorems for a particle in an electromagnetic field

J I Jiménez-Aquino, F J Uribe and R M Velasco

Departamento de Física, Universidad Autónoma Metropolitana–Iztapalapa, Apartado Postal 55–534, CP 09340, México, Distrito Federal, Mexico

E-mail: ines@xanum.uam.mx

Received 2 March 2010, in final form 21 April 2010
Published 26 May 2010
Online at stacks.iop.org/JPhysA/43/255001

Abstract
The theoretical study about the transient and stationary fluctuation theorems is extended to include the effects of electromagnetic fields on a charged Brownian particle. In particular, we consider a harmonic trapped Brownian particle under the action of a constant magnetic field pointing perpendicular to a plane and a time-dependent electric field acting on this plane. The electric field is seen to be responsible for the motion of the center in the harmonic trap, giving as a result a time-dependent dragging. Our study is focused on the solution of the Smoluchowski equation associated with the over-damped Langevin equation and also considers two particular cases for the motion of the harmonic trap minimum. The first one is produced by a linear time-dependent electric field and, in the second case an oscillating electric field produces a circular motion. In this last case we have found resonant behavior in the mean work when the electric field is tuned with Larmor's frequency. Some comparisons are made with other works in the absence of the magnetic field.

PACS number: 05.40.-a

1. Introduction

Fluctuation theorems (FTs) and related research have been of great interest in nonequilibrium statistical physics of small systems in which the fluctuations play a fundamental role [1–47]. Much of the work done in developing and extending the theorems was accomplished by theoreticians interested in nonequilibrium statistical mechanics. These theorems involve a wide class of systems as well as several equilibrium and nonequilibrium quantities, including the Helmholtz free energy [3], work [14], heat [15] and entropy production [21]. They can be applied to steady state situations [14, 22] and the transient theorems allow us to go a step further [14, 20, 21]. The FTs have been corroborated by both computer simulations [20, 24–26, 29, 36] and experiments [26–29]. The first experiment performed to demonstrate the FT was reported by Wang *et al* [26], for a small system over short times. In this experiment the trajectory of a colloidal particle is followed when it is captured in an optical trap that is translated relative to surrounding water molecules. The experiment confirms a theoretically predicted work relation associated with the integrated transient fluctuation theorem (ITFT), which gives an expression for the ratio between the probability of finding the positive and negative values of the fluctuations in the total work done on the system for a given time in a transient state. However, the integrated stationary state fluctuation theorem (ISSFT) was not observed. After the experiment by Wang *et al* [26] others were continued, for example with colloidal particles in harmonic trap potentials [27, 28]. On the other hand, Blickle *et al* [29] conducted another experiment to study an over-damped colloidal particle in a time-dependent nonharmonic potential. They have shown the validity of a balance between work, heat and energy, which looks like the first law applied to a stochastic trajectory. Also, the Jarzinski equality (JE) and the work-fluctuation theorem are valid for this case. It was also shown that nonharmonic potentials give rise to non-Gaussian work distributions. This fact confirms not only the validity of FTs for Gaussian systems, white and colored noise [5], but also for those situations where the underlying distributions are typically non-Gaussian. When fluctuations are due to Lévy noise [6] or Poissonian shot noise [7], it has been proved that the stationary state fluctuation theorem (SSFT) does not hold.

Inspired by Wangs *et al*'s experiment [26], and using a model of a Brownian particle in a harmonic potential with a minimum moving arbitrarily, van Zon and Cohen (vZC) [14] showed theoretically that all quantities of interest for these theorems and the corresponding SSFT and the transient fluctuation theorem (TFT) hold. Our goal in this work is concentrated on the generalization of such theorems to the case in which the Brownian harmonic oscillator is electrically charged and it is in the presence of an electromagnetic field, assuming the action of a Gaussian white noise. The magnetic field is considered as a constant and pointing along the z-axis, and the electric field is considered as a space-independent but in general a time-dependent vector which is responsible for the dragging of the potential minimum in an arbitrary way. To achieve our goal we solve the Smoluchowski equation (SE) associated with the charged Brownian harmonic oscillator in an electromagnetic field, by means of a change of variable, defined by the $\mathbf{X}(t)$ variable, which involves a rotation matrix to account for the effects of the magnetic field. The transition probability density associated with the fluctuating variable $\mathbf{X}(t)$ is calculated explicitly for all time $t > 0$. Also, it is shown that it has the Gaussian stationary distribution. Under these conditions we verify both the TFT and the SSFT, whereas their integrated versions will only be commented.

It is important to note here that FTs including harmonic trap potentials have also been verified recently for a swarm of independent Brownian harmonic oscillators in the presence of an electromagnetic field [47]. In this reference, the somewhat unexpected Hall-type fluctuation relation, firstly established by Roy and Kumar [11], is obtained as a particular case. FTs have also been extended to more complex systems, such as colloidal particles trapped in harmonic traps [26] and certain models for polymers [17, 18]. On the other hand, in the case of nonlinear potentials one is not always able to give explicit solutions and therefore numerical methods are very likely to be used [29], in this case the explicit solutions obtained for linear problems can be used as an aid to validate the numerical computations. In fact, recent numerical methods, expected to be used in the field of complex (dusty) plasmas [30] (a well-established subdiscipline of plasma physics [31]), have been validated by comparing them with the available analytical solutions for a Brownian harmonic oscillator in an external magnetic field [32]. In the field of plasma physics, other problems have been studied in the context of the Brownian motion [33, 34] and the FTs [35]. In the last work, Consolini *et al* [35] have proved the validity of the FT for a problem related to the Earth's magnetosphere, which evolves as an out-of-equilibrium system due to the coupling with the solar wind and the Earth's ionosphere. Although in this work we are not concerned with quantum considerations, it is worth mentioning that the interest in the fluctuation relations is now included in the quantum description [48–50].

The other quantity related to the work-fluctuation theorems is the JE, which relates averages of non-equilibrium quantities to the equilibrium free-energy differences between equilibrium states [3]. This equality has been verified experimentally in biological systems [37]. The generalizations to arbitrary transitions between non-equilibrium stationary states [42, 43] have also been verified in the experiment [28]. Recently, TFT has been proved for the Brownian motion of a classical harmonic oscillator under the action of a magnetic field [44–46], and the JE has been used in [44, 46] to show its consistency with the Bohr–van Leeuwen (BvL) theorem in the absence of orbital diamagnetism in a classical system of charged particles in thermodynamic equilibrium [51]. However, to the best of our knowledge, the fluctuation relations and the JE for a charged harmonic oscillator in an electromagnetic field have not been tested experimentally. Thus, our present results might in turn motivate experimentalists to perform new experiments.

Our work is then structured as follows. In section 2, we introduce the complete Langevin equation for the charged harmonic oscillator in an electromagnetic field and establish the required conditions for the parameters to obtain the over-damped Langevin equation. Section 3 is devoted to the definition of total work and we give the mathematical conditions to calculate its properties. In section 4, we prove the validity of the TFT when the charged Brownian particle is dragged in an arbitrary way by the time-dependent electric field. In section 5, we give the proof for the SSFT and comment on the conditions under which the integrated version of both theorems are valid. Two physical situations for the movement of the potential's minimum are studied in section 6. In the first case the minimum is dragged with a linear constant velocity driven by the electric field, while in the second case this minimum is forced to move by an oscillating electric field. In both cases, our theoretical results will be compared with those calculated in [14], in the absence of the magnetic field. As a final application, we use the TFT and JE to show that between two equilibrium states there is no free energy differences, consistently with the BvL theorem. Our conclusions are given in section 7.

2. The Langevin equation

Let us consider a charged Brownian particle with mass m and charge q in a harmonic trap with a center driven by an external electric field $\mathbf{E}(t)$. The electric field is homogeneous but it can be time dependent. Besides, a uniform magnetic field pointing along the z-axis is acting upon it. The surrounding medium is at a temperature T and it produces a fluctuating force $\mathbf{f}(t) = (f_x, f_y, f_z)$, which satisfies the properties of Gaussian white noise with the zero mean value $\langle f_i(t)\rangle = 0$ and a correlation function given by

$$\langle f_i(t) f_j(t')\rangle = 2\lambda\delta_{ij}\delta(t - t'), \tag{1}$$

with $i, j = x, y, z$ and λ being a constant which measures the noise intensity and according to the fluctuation–dissipation theorem is related to the friction constant by $\lambda = \gamma\, k_B T$, k_B being Boltzmann's constant. The Langevin equation corresponding to this system is then given as

$$m\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = -\gamma\mathbf{v} + \frac{q}{c}\mathbf{v}\times\mathbf{B} - k\mathbf{r} + q\mathbf{E}(t) + \mathbf{f}(t), \tag{2}$$

where $\mathbf{r}$ is the position vector of the particle, γ is the friction coefficient and k is the restitutive constant in the harmonic potential. The Langevin equation, as written in equation (2), represents Newton's second law for a Brownian particle. In fact, it contains the systematic force given by the Stokes friction, the harmonic force and the Lorentz force for a charged particle. Besides, the fluctuating force $\mathbf{f}(t)$ takes into account the interaction of the Brownian particle with the fluid in which it is immersed. Such an equation has a phenomenological support which has been extensively used in the past. By the way, it is important to comment that the first phenomenological study on diffusion of plasma in the crossed magnetic field was performed by Taylor [56], in the context of the Langevin equation by assuming a fluctuating electric field. Next, Kurşunoğlu [57] could extend the study using a description in terms of probability densities. After these studies, several works on the same topic have arisen using some other mathematical tools [30, 33, 58–60]. To the best of our knowledge, equation (2) has not been derived from Hamiltonian dynamics as it is actually the case in the absence of the magnetic field [61–63]. In equation (2), the electric field can be interpreted as responsible for dragging the center of the harmonic trap; hence if we denote the position vector of the center as $\mathbf{r}^*$ in the harmonic trap, it can be written as $\mathbf{r}^* = (q/k)\mathbf{E}(t)$. For $t = 0$, the potential minimum is located at the origin, $\mathbf{r}_0^* = 0$, whereas for $t > 0$, it moves with a velocity $\mathbf{v}^*(t)$ driven by the electric field in such a way that $\dot{\mathbf{r}}(t) \equiv \mathbf{v}^*(t) = (q/k)\dot{\mathbf{E}}(t)$. Now the Langevin equation is written as follows:

$$m\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = -\gamma\mathbf{v} + \frac{q}{c}\mathbf{v}\times\mathbf{B} - k(\mathbf{r}-\mathbf{r}^*) + \mathbf{f}(t). \tag{3}$$

Due to the fact that the magnetic field is pointing along the z-axis, equation (3) can be decomposed into two independent differential equations, one is given on the x–y plane perpendicular to the magnetic field and the other one along the z-axis parallel to the magnetic field. Along the z-axis, the Langevin equation is the same z-component of the Langevin equation studied in [14]. In this work, we will pay attention on the over-damped planar Langevin equation, for which we first define the following quantities on the x–y plane: $\mathbf{x}$ as the position vector, $\mathbf{u}$ the velocity vector, $\bar{\mathbf{f}}(t) = (f_x, f_y)$ the fluctuating force and $\bar{\mathbf{E}}(t)$ the electric field. Also, the two-dimensional harmonic potential $\bar{U}$ and its corresponding harmonic force $\bar{\mathbf{F}}$ are given by

$$\bar{U}(\mathbf{x}, t) = \frac{k}{2}|\mathbf{x}-\mathbf{x}^*|^2, \qquad \bar{\mathbf{F}}(\mathbf{x}, \mathbf{x}^*) = -k(\mathbf{x}-\mathbf{x}^*), \tag{4}$$

where $\mathbf{x}^* = (q/k)\bar{\mathbf{E}}(t)$ and $\dot{\mathbf{x}}^* = \mathbf{u}^*(t) = (q/k)\dot{\bar{\mathbf{E}}}(t)$. In this case the dragging of the potential's minimum depends on the electric field's rate of change. The over-damped approximation is satisfied when the parameters appearing in the Langevin equation, equation (3), satisfy $\omega^2 \ll \rho^2[1 + (\Omega/\rho)^2]$, where $\Omega = qB/mc$ is Larmor's frequency and $\rho = \gamma/m$. This condition is equivalent to $km \ll \gamma_e^2$, where $\gamma_e = \gamma(1 + C^2)$ with $C \equiv \Omega/\rho = qB/c\gamma$ being a dimensionless quantity. It is clear that γ_e plays the role of an effective friction coefficient and it is clearly magnetic field dependent. Obviously, this condition reduces to the usual one when the magnetic field is absent [14]. Therefore, in the over-damped approximation the planar Langevin equation reads

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = -\widetilde{\gamma}\mathbf{x} - \widetilde{W}\mathbf{x} + \Lambda\mathbf{x}^* + k^{-1}\Lambda\bar{\mathbf{f}}(t), \tag{5}$$

where $\widetilde{W}$ and $\Lambda = \widetilde{\gamma}\mathbf{I} + \widetilde{W}$ are given by

$$\widetilde{W} = \begin{pmatrix} 0 & \widetilde{\Omega} \\ -\widetilde{\Omega} & 0 \end{pmatrix}, \qquad \Lambda = \begin{pmatrix} \widetilde{\gamma} & \widetilde{\Omega} \\ -\widetilde{\Omega} & \widetilde{\gamma} \end{pmatrix}, \tag{6}$$

$\mathbf{I}$ being the unit matrix, $\widetilde{\gamma} = k/\gamma(1 + C^2) = k/\gamma_e$ and $\widetilde{\Omega} = kC/\gamma(1 + C^2)$.

Next, we separate the average motion of the charged Brownian particle from the stochastic motion. For the average motion we propose the variable $\mathbf{y}^*$ as the deterministic solution of equation (5), that is

$$\frac{\mathrm{d}\mathbf{y}^*}{\mathrm{d}t} = -\widetilde{\gamma}\mathbf{y}^* - \widetilde{W}\mathbf{y}^* + \Lambda\mathbf{x}^*, \tag{7}$$

with the initial condition $\mathbf{y}^*(0) = \mathbf{y}_0^* = 0$. Now, we introduce the fluctuating variable $\mathbf{X} = \mathbf{x} - \mathbf{y}^*$, for which the Langevin equation reads

$$\frac{\mathrm{d}\mathbf{X}}{\mathrm{d}t} = -\widetilde{\gamma}\mathbf{X} - \widetilde{W}\mathbf{X} + k^{-1}\Lambda\bar{\mathbf{f}}(t). \tag{8}$$

As we will see later on, we require the stationary probability density for the fluctuating variable $\mathbf{X}$. However, we can go further because we can calculate not only this stationary probability but also the probability density for all time $t > 0$, through the explicit solution of the Fokker–Planck (FP) equation associated with the fluctuating variable. The strategy of the solution is given in terms of the following change of variable [47]:

$$\mathbf{X}' = \mathrm{e}^{\widetilde{W}t}\mathbf{X} = \mathbf{x}' - \mathbf{y}'^*, \tag{9}$$

where $\mathbf{x}' = \mathrm{e}^{\widetilde{W}t}\mathbf{x}$, $\mathbf{y}'^* = \mathrm{e}^{\widetilde{W}t}\mathbf{y}^*$, and $\mathcal{R}(t) = \mathrm{e}^{\widetilde{W}t}$ is an orthogonal rotation matrix given by

$$\mathcal{R}(t) = \begin{pmatrix} \cos\widetilde{\Omega}t & \sin\widetilde{\Omega}t \\ -\sin\widetilde{\Omega}t & \cos\widetilde{\Omega}t \end{pmatrix}, \tag{10}$$

and its inverse is given by $\mathcal{R}^{-1}(t) = \mathrm{e}^{-\widetilde{W}t}$. Clearly in the new space of coordinates we have

$$\frac{\mathrm{d}\mathbf{y}'^*}{\mathrm{d}t} = -\widetilde{\gamma}\mathbf{y}'^* + \Lambda\mathbf{x}'^*, \tag{11}$$

$$\frac{\mathrm{d}\mathbf{X}'}{\mathrm{d}t} = -\widetilde{\gamma}\mathbf{X}' + \bar{\mathbf{f}}'(t), \tag{12}$$

with $\bar{\mathbf{f}}'(t) = k^{-1}\Lambda\mathcal{R}(t)\bar{\mathbf{f}}(t)$. The stochastic variable $\mathbf{X}'$ satisfies the standard Ornstein–Uhlenbeck process [52, 54] because the transformed noise $\bar{\mathbf{f}}'(t)$ satisfies the same statistical properties of Gaussian white noise than the original noise $\bar{\mathbf{f}}(t)$. The associated FP equation for the transition probability density $P'(\mathbf{X}', t|\mathbf{X}_0')$ is [47]

$$\frac{\partial P'}{\partial t} = \widetilde{\gamma}\,\mathbf{div}_{\mathbf{X}'}(\mathbf{X}'P') + \widetilde{\lambda}\nabla_{\mathbf{X}'}^2 P', \tag{13}$$

with $\widetilde{\lambda} = \lambda/\gamma^2(1 + C^2)$. The solution to equation (13) is well known [47, 53–55], and reads

$$P'(\mathbf{X}', t|\mathbf{X}_0') = \frac{\beta k}{2\pi(1 - \mathrm{e}^{-2\widetilde{\gamma}t})}\exp\left(-\frac{\beta k|\mathbf{X}' - \mathrm{e}^{-\widetilde{\gamma}t}\mathbf{X}_0'|^2}{2(1 - \mathrm{e}^{-2\widetilde{\gamma}t})}\right), \tag{14}$$

where $P'(\mathbf{X}', 0|\mathbf{X}_0') = \delta(\mathbf{X}' - \mathbf{X}_0')$. According to transformation (9), the transition probability density for the fluctuating variable $\mathbf{X}$ for all time $t > 0$ can readily be shown to be

$$P(\mathbf{X}, t|\mathbf{X}_0) = \frac{\beta k}{2\pi(1 - \mathrm{e}^{-2\widetilde{\gamma}t})}\exp\left(-\frac{\beta k|\mathbf{X} - \mathrm{e}^{-\Lambda t}\mathbf{X}_0|^2}{2(1 - \mathrm{e}^{-2\widetilde{\gamma}t})}\right). \tag{15}$$

Clearly, the corresponding stationary distribution is given by $P_{\mathrm{eq}}(\mathbf{X}) = (\beta k/2\pi)\,\mathrm{e}^{-(\beta k/2)|\mathbf{X}|^2}$. In a similar way, it is possible to consider the initial conditions as a Gaussian distribution given by $P(\mathbf{X}_0) = (\beta k/2\pi)\mathrm{e}^{-(\beta k/2)|\mathbf{X}_0|^2}$, though the explicit calculation is not given.

3. Transient fluctuation theorem for the total work

The work definition has given place to some difficulties because the total work contains some terms responsible for the change in mechanical energy. In this work, we adopt the definition of the dimensionless total work done on a system during a time τ as given in [14]:

$$\mathbb{W}_\tau^{\text{tot}} = \frac{1}{k_B T}\int_0^\tau \mathbf{u}^* \cdot \mathbf{F}(\mathbf{x}, \mathbf{x}^*)\, \mathrm{d}t. \tag{16}$$

According to this definition and the developments done above, we calculate the statistical properties of the total dimensionless work for the harmonic force defined in equation (4), which in terms of the $\mathbf{X}$ variable (defined above equation (8)) can be written as

$$\mathbb{W}_\tau = -\beta k \int_0^\tau [\mathbf{u}^* \cdot \mathbf{X} + \mathbf{u}^* \cdot (\mathbf{y}^* - \mathbf{x}^*)]\, \mathrm{d}t, \tag{17}$$

where $\beta = 1/(k_B T)$. Equation (17) shows that the total work done on the system is a linear function of the stochastic variable $\mathbf{X}$. Accordingly, the statistical properties of the dimensionless total work correspond to a Gaussian process. Therefore the probability distribution P_T of the total work can be written as follows:

$$P_T(\mathbb{W}_\tau) = \frac{1}{\sqrt{2\pi V_T(\tau)}}\exp\left(-\frac{[\mathbb{W}_\tau - M_T(\tau)]^2}{2V_T(\tau)}\right), \tag{18}$$

where we have defined $M_T(\tau) \equiv \langle \mathbb{W}_\tau \rangle$ as the mean value of the work and $V_T(\tau) \equiv \langle \mathbb{W}_\tau^2 \rangle - \langle \mathbb{W}_\tau \rangle^2$ as its variance. The probability distribution written in equation (18) contains the time evolution of the total work from the initial time up to time τ. This fact means that we are studying the distribution corresponding to the transient situation. We will use the subscript T for all quantities in the transient case. Taking into account that $\langle \mathbf{X} \rangle = 0$, the work mean value reads

$$M_T(\tau) = -\beta k \int_0^\tau \mathbf{u}^* \cdot (\mathbf{y}^* - \mathbf{x}^*)\, \mathrm{d}t. \tag{19}$$

On the other hand, the variance is only affected by the first term in equation (17), so that

$$V_T(\tau) = (\beta k)^2 \int_0^\tau \mathrm{d}t_1 \int_0^\tau \mathbf{u}^*(t_1) \cdot \langle \mathbf{X}(t_1)\mathbf{X}(t_2) \rangle \cdot \mathbf{u}^*(t_2)\, \mathrm{d}t_2. \tag{20}$$

In order to calculate the work mean value we need the solution for the $\mathbf{y}^*$ variable, which can be calculated from (11):

$$\mathbf{y}'^*(t) = \mathrm{e}^{-\widetilde{\gamma} t}\mathbf{y}_0'^* + \int_0^t \mathrm{e}^{-\widetilde{\gamma}(t-t')}\Lambda \mathbf{x}'^*(t')\, \mathrm{d}t' = \mathrm{e}^{-\widetilde{\gamma} t}\mathbf{y}_0'^* + \mathrm{e}^{-\widetilde{\gamma} t}\int_0^t \mathrm{e}^{\Lambda t'}\Lambda \mathbf{x}^*(t')\, \mathrm{d}t', \tag{21}$$

where we made use of the relation $\Lambda = \widetilde{\gamma}\,\mathbf{I} + \widetilde{W}$ and $\mathcal{R}(t) = \mathrm{e}^{\widetilde{W} t}$. Since $\mathbf{y}_0^* = 0$ then $\mathbf{y}_0'^* = 0$ and after integration by parts, equation (21) reduces to

$$\mathbf{y}'^*(t) = \mathbf{x}'^*(t) - \int_0^t \mathrm{e}^{-\widetilde{\gamma}(t-t')}\mathcal{R}(t')\mathbf{u}^*(t')\, \mathrm{d}t', \tag{22}$$

and therefore the solution for the $\mathbf{y}^*$ variable reads

$$\mathbf{y}^*(t) = \mathbf{x}^*(t) - \mathcal{R}^{-1}(t)\int_0^t \mathrm{e}^{-\widetilde{\gamma}(t-t')}\mathcal{R}(t')\mathbf{u}^*(t')\, \mathrm{d}t'. \tag{23}$$

By substituting equation (23) into equation (19) and after some algebra we can show that this work mean value can be written as

$$M_T(\tau) = \beta k \int_0^\tau \mathrm{d}t' \int_0^{t'} \mathrm{e}^{-\widetilde{\gamma}(t'-t'')}\mathbf{U}^*(t') \cdot \mathbf{U}^*(t'')\, \mathrm{d}t'', \tag{24}$$

where we have defined $\mathbf{U}^*(t) = \mathcal{R}(t)\mathbf{u}^*(t)$, which accounts for a rotation of the velocity $\mathbf{u}^*$. To evaluate the variance we take into account the symmetry of the time-correlation function $\langle \mathbf{X}(t_1)\mathbf{X}(t_2)\rangle$ under the interchange of t_1 and t_2; hence equation (20) can be written as

$$V_T(\tau) = 2(\beta k)^2 \int_0^\tau \mathrm{d}t' \int_0^{t'} \mathbf{u}^*(t') \cdot \langle \mathbf{X}(t')\mathbf{X}(t'')\rangle \cdot \mathbf{u}^*(t'')\,\mathrm{d}t''. \tag{25}$$

To continue the calculation we need the time correlation $\langle \mathbf{X}(t')\mathbf{X}(t'')\rangle$, which corresponds to a stationary process, so that this time correlation can also be written as $\langle \mathbf{X}(t'-t'')\mathbf{X}_0\rangle$. Also, the solution for this variable can be obtained from equations (8) and (12) with the properties of the matrix $\mathcal{R}(t)$, from which for all $t \geqslant 0$, it can be written as

$$\mathbf{X}(t) = \mathrm{e}^{-\widetilde{\gamma}t}\mathcal{R}^{-1}(t)\mathbf{X}_0 + \mathcal{R}^{-1}(t)\int_0^t \mathrm{e}^{-\widetilde{\gamma}(t-t')}\bar{\mathbf{f}}'(t')\,\mathrm{d}t'. \tag{26}$$

Since $\langle \mathbf{X}_0\mathbf{X}_0\rangle = (k_B T/k)\mathbf{I}$ and if $\langle \bar{\mathbf{f}}(t)\mathbf{X}_0\rangle = 0$, we can show that for $t \geqslant 0$

$$\langle \mathbf{X}(t)\mathbf{X}_0\rangle = (\beta k)^{-1}\,\mathrm{e}^{-\widetilde{\gamma}t}\mathcal{R}^{-1}(t)\mathbf{I}, \tag{27}$$

which implies that for $t' \geqslant t''$,

$$\langle \mathbf{X}(t'-t'')\mathbf{X}_0\rangle = (\beta k)^{-1}\,\mathrm{e}^{-\widetilde{\gamma}(t'-t'')}\mathcal{R}^{-1}(t')\mathcal{R}(t'')\mathbf{I}. \tag{28}$$

With the direct substitution of equation (28) into equation (25) it can be verified that the variance yields

$$V_T(\tau) = 2\beta k \int_0^\tau \mathrm{d}t' \int_0^{t'} \mathrm{e}^{-\widetilde{\gamma}(t'-t'')}\mathbf{U}^*(t') \cdot \mathbf{U}^*(t'')\,\mathrm{d}t'', \tag{29}$$

and the comparison with equation (24) allows us to conclude that

$$V_T(\tau) = 2M_T(\tau). \tag{30}$$

Then, according to equations (18) and (30), we can write the ratio of the probability distributions $P_T(\mathbb{W}_\tau)$ and $P_T(-\mathbb{W}_\tau)$ as follows:

$$\frac{P_T(\mathbb{W}_\tau)}{P_T(-\mathbb{W}_\tau)} = \mathrm{e}^{2M_T(\tau)\mathbb{W}_\tau/V_T(\tau)} = \mathrm{e}^{\mathbb{W}_\tau}. \tag{31}$$

Therefore, the TFT for the dragging of an electrically Brownian charged harmonic oscillator in the presence of a uniform magnetic field and a time-varying electric field is also satisfied in this case.

4. Stationary state fluctuation theorem for the total work

In this section we will consider the SSFT when an electromagnetic field is present. The SSFT is formulated for the dimensionless total work done on the system, during the time interval τ along a single trajectory in the stationary state. First of all, we consider the total work given in equation (16) for the plane perpendicular to the magnetic field. In this case the total dimensionless work done on the system over the time interval τ will be given as

$$\mathbb{W}_{\tau,t_i} = \beta \int_{t_i}^{t_i+\tau} \mathbf{F}(\mathbf{x}, \mathbf{x}^*) \cdot \mathbf{u}^*\,\mathrm{d}t. \tag{32}$$

We emphasize that this work will be calculated for a sequence of initial times t_i and all segments correspond to a time interval τ calculated along a single stationary state trajectory

($i = 1, 2, 3, \ldots$). As a second step we recall that the stochastic variable $\mathbf{X}$ being a Gaussian process has a Gaussian stationary state. This fact allows us to assure that the total work will have the same property; therefore, the distribution of $\mathbb{W}_{\tau,t_i}$ for each t_i is also Gaussian and given by

$$P_{t_i}(\mathbb{W}_{\tau,t_i}) = \frac{1}{\sqrt{2\pi V_{t_i}(\tau)}} \exp\left(- \frac{[\mathbb{W}_{\tau,t_i} - M_{t_i}(\tau)]^2}{2V_{t_i}(\tau)} \right), \tag{33}$$

where the mean value and the variance are respectively given by

$$M_{t_i}(\tau) = -\beta k \int_{t_i}^{t_i+\tau} \mathbf{u}^* \cdot (\mathbf{y}^* - \mathbf{x}^*)\, \mathrm{d}t, \tag{34}$$

$$V_{t_i}(\tau) = 2\beta k \int_{t_i}^{t_i+\tau} \mathrm{d}t' \int_{t_i}^{t'} \mathrm{e}^{-\widetilde{\gamma}(t'-t'')} \mathbf{U}^*(t') \cdot \mathbf{U}^*(t'')\, \mathrm{d}t''. \tag{35}$$

We also assume that for each t_i large enough, M_{t_i} and V_{t_i} will reach steady state values, and consequently become independent of t_i. If in addition, the correlation between different segments $[t_i, t_i + \tau]$ and $[t_j, t_j + \tau]$ decays sufficiently fast as $|t_i - t_j|$ gets larger, the distribution of W_{τ,t_i} along a trajectory in the stationary state is given by

$$P_S(\mathbb{W}_{\tau,S}) = \frac{1}{\sqrt{2\pi V_S(\tau)}} \exp\left(- \frac{[\mathbb{W}_{\tau,S} - M_S(\tau)]^2}{2V_S(\tau)} \right), \tag{36}$$

where the subscript S denotes the distribution of $\mathbb{W}_{\tau,t_i} \to \mathbb{W}_{\tau,S}$ over segments along the stationary state trajectory. Thus for large time and according to equations (23) and (34), the mean value is

$$M_S(\tau) = \lim_{t\to\infty} \beta k \int_t^{t+\tau} \mathrm{d}t' \int_0^{t'} \mathrm{e}^{-\widetilde{\gamma}(t'-t'')} \mathbf{U}^*(t') \cdot \mathbf{U}^*(t'')\, \mathrm{d}t'', \tag{37}$$

and according to equation (35), the variance reads

$$V_S(\tau) = \lim_{t\to\infty} 2\beta k \int_t^{t+\tau} \mathrm{d}t' \int_t^{t'} \mathrm{e}^{-\widetilde{\gamma}(t'-t'')} \mathbf{U}^*(t') \cdot \mathbf{U}^*(t'')\, \mathrm{d}t''. \tag{38}$$

In equations (37)–(38), the integration limits make a difference when compared with equations (24) and (29). This difference is manifested when we realized that the equality between V_S and $2M_S$ is not satisfied, in contrast with the transient case, where we have obtained that $V_T = 2M_T$. The corresponding deviation can be calculated from

$$\frac{P_S(\mathbb{W}_\tau)}{P_S(-\mathbb{W}_\tau)} = \exp\left(\frac{\mathbb{W}_\tau}{1 - \varepsilon(\tau)} \right), \tag{39}$$

where $\varepsilon(\tau)$ represents the deviation between the mean value and the variance, and it is given by

$$\varepsilon(\tau) = \frac{2M_S(\tau) - V_S(\tau)}{2M_S(\tau)}. \tag{40}$$

The direct substitution of equations (37)–(38) in equation (40) leads us to the following expression for the quantity $\varepsilon(\tau)$:

$$\varepsilon(\tau) = \frac{1}{M_S(\tau)} \lim_{t\to\infty} \beta k \int_t^{t+\tau} \mathrm{d}t' \int_0^{t} \mathrm{e}^{-\widetilde{\gamma}(t'-t'')} \mathbf{U}^*(t') \cdot \mathbf{U}^*(t'')\, \mathrm{d}t'', \tag{41}$$

which according to equation (23) can be written as

$$\varepsilon(\tau) = \frac{1}{M_s(\tau)} \lim_{t\to\infty} \beta k(\mathbf{x}^* - \mathbf{y}^*) \cdot \int_0^\tau \mathrm{e}^{-\widetilde{\gamma} t'} \mathbf{U}^*(t + t')\,\mathrm{d}t'. \tag{42}$$

In equation (42) the denominator corresponds to the total dimensionless work done of the system in the stationary state in time τ. In the numerator, the exponential in the integral will make the integral bounded for large time τ, provided that $\mathbf{U}^*(t)$ does not grow exponentially in time with an exponent bigger than the relaxation time $\widetilde{\tau}_r = \widetilde{\gamma}^{-1}$. We note that this relaxation time depends on the magnetic field. Then $\varepsilon(\tau)$ approaches to zero proportionally to $1/\tau$ as τ goes to infinity; hence

$$\varepsilon(\tau) \to 0 \qquad \text{as} \qquad \tau \to \infty. \tag{43}$$

As a consequence we obtain that

$$V_s(\tau) \to 2M_s(\tau) \qquad \text{as} \qquad \tau \to \infty, \tag{44}$$

and the SSFT holds, that is

$$\frac{P_s(\mathbb{W}_{\tau,s})}{P_s(-\mathbb{W}_{\tau,s})} \to \mathrm{e}^{\mathbb{W}_{\tau,s}} \qquad \text{as} \qquad \tau \to \infty. \tag{45}$$

Once we have proved the TFT and SSFT given respectively by equations (31) and (45), we can prove the integrated version of these theorems following similar steps as established in [14]. It is important to note that the magnetic field can drive to different effects, though the formal structure is similar.

5. Applications

Here we will study two cases in which the two-dimensional harmonic trap potential is moving. In the first case, we assume that the electric field depends linearly on time $\mathbf{E}(t) = (\mathcal{E}t, \mathcal{E}t)$, so the potential minimum is dragged with uniform velocity (linear motion). This is the same kind of motion studied by Jayannavar and Sahoo [44] to verify the BvL theorem [51] using the JE. As a second application, we will consider an oscillating electric field which produces a circular motion in the minimum of the harmonic trap [14]. Both physical models can be used in principle by experimentalists to corroborate the work-fluctuation theorems, as has been done in the absence of a magnetic field.

5.1. Linear electric field in the transient case

In this first case, the position vector for the potential minimum can be written as $\mathbf{x}^* = (q/k)\mathcal{E}t = (ut, ut)$, meaning that the electric field drags the minimum of the trap with a constant velocity $\mathbf{u}^* = u_{\mathrm{opt}}(1, 1)$, where $u_{\mathrm{opt}} = (q/k)\mathcal{E}$ is the so-called optical speed and $\mathcal{E}$ is the amplitude of the electric field per unit time. The calculations for the total work mean value and its variance are explicitly given in appendix A. In this case both quantities are given respectively by

$$M_T(\tau) = 2\varpi\{\tau - \tau_r(1 - C^2)[1 - \mathrm{e}^{-\tau/\widetilde{\tau}_r} \cos(\widetilde{\Omega}\tau)] - 2\tau_r C\, \mathrm{e}^{-\tau/\widetilde{\tau}_r} \sin(\widetilde{\Omega}\tau)\}, \tag{46}$$

$$V_T(\tau) = 2M_T(\tau), \tag{47}$$

where $\varpi = \beta\gamma\, u_{\mathrm{opt}}^2 = \beta\gamma(q\mathcal{E})^2/k^2$ stands as the dimensionless work delivered to the system per unit time. Equation (46) multiplied by the factor β^{-1} is the total work mean value defined in equation (16), and it is exactly the same as the one calculated by Jayannavar and Sahoo in [44], by an alternative method. In the absence of the magnetic field ($C = 0$), the transient dimensionless work mean value reduces to $M_{_T}(\tau) = 2\varpi\{\tau - \tau_r\,[1 - \mathrm{e}^{-k\tau/\gamma}]\}$, which is the same expression calculated by van Zon and Cohen [14] and Mazonka and Jarzynski [19] except by the factor 2 which comes from the planar character of the vector $\mathbf{u}^*$.

5.2. Linear electric field in the stationary case

In appendix A we have shown that the work mean value and its variance read

$$M_{_S}(\tau) = 2\varpi\tau, \tag{48}$$

$$V_{_S}(\tau) = V_T(\tau) = 2M_{_T}(\tau). \tag{49}$$

On the other hand, the deviation $\varepsilon(\tau)$ defined in equation (40) is given by

$$\varepsilon(\tau) = \frac{1}{\tau}\{\tau_r(1 - C^2)[1 - \mathrm{e}^{-\tau/\widetilde{\tau}_r}\cos(\widetilde{\Omega}\tau)] + 2C\tau_r\,\mathrm{e}^{-\tau/\widetilde{\tau}_r}\sin(\widetilde{\Omega}\tau)\}, \tag{50}$$

which is proportional to $1/\tau$ and it is clear that $\varepsilon(\tau) \to 0$ as $\tau \to \infty$. Thus, according to equation (44) we have that $V_{_S}(\tau) = 2M_{_S}(\tau)$ and consequently the SSFT holds. In addition, this equality can be obtained from equations (48) and (49) because from equation (47) we see that for large τ, $M_{_T}(\tau)$ is proportional to τ and therefore $M_{_T}(\tau) = M_{_S}(\tau)$.

5.3. Oscillating electric field in the transient case

As a second example, let us consider an oscillating electric field driving the potential minimum into a circular motion. Now, its position vector is given as $\mathbf{x}^*(t) = r(\sin\Omega_0 t, (1-\cos\Omega_0 t))$ for $t \geqslant 0$, $r = qE/k$ is the radius of the circle and E is the amplitude of the electric field taken as a constant. Thus, the dragging velocity is $\mathbf{u}^*(t) = u_{\rm opt}(\cos\Omega_0 t, \sin\Omega_0 t)$, and $u_{\rm opt} = r\Omega_0$ is the corresponding optical speed. In appendix B the transient mean value of the total dimensionless work is explicitly calculated. Its value and the corresponding variance are given as follows:

$$M_{_T}(\tau) = \varpi_e\left\{\tau - \widetilde{\tau}_r\frac{2\widetilde{\tau}_r\widehat{\Omega}\,\mathrm{e}^{-\tau/\widetilde{\tau}_r}\sin(\widehat{\Omega}\tau)}{1 + \widetilde{\tau}_r^2\widehat{\Omega}^2} - \widetilde{\tau}_r\frac{\left[1 - \widetilde{\tau}_r^2\widehat{\Omega}^2\right][1 - \mathrm{e}^{-\tau/\widetilde{\tau}_r}\cos(\widehat{\Omega}\tau)]}{1 + \widetilde{\tau}_r^2\widehat{\Omega}^2}\right\}, \tag{51}$$

$$V_{_T}(\tau) = 2M_{_T}(\tau), \tag{52}$$

where ϖ_e is the work per unit time given by

$$\varpi_e = \frac{\beta\gamma_e u_{\rm opt}^2}{(1 + \widetilde{\tau}_r^2\widehat{\Omega}^2)}, \tag{53}$$

and $\widehat{\Omega} = \Omega_0 - \widetilde{\Omega}$. At this point it is very interesting to note that in the special case where $\Omega_0 = \widetilde{\Omega}$, the mean value of the total work has a maximum given by $M_{_T}(\tau) = \beta\gamma_e u_{\rm opt}^2[\tau - \widetilde{\tau}_r(1 - \mathrm{e}^{-\tau/\widetilde{\tau}_r})]$. In fact, in figures 1 and 2, we show the symmetric behavior around the maximum of a reduced value of (51), that is $M_{_T} \equiv M_{_T}(\tau)/\widetilde{\tau}_r\beta\gamma_e\,u_{\rm opt}^2$, as a function of the dimensionless variables $x = \tau/\widetilde{\tau}_r$ and $y = \widehat{\Omega}\,\widetilde{\tau}_r$. Thus, the maximum means that we have a resonance caused by the tuning in the electric field with the Larmor frequency. In other words we have a case where both the magnetic and the oscillating electric field are tuned, giving place to a resonant situation which can be explored by an experimental device.

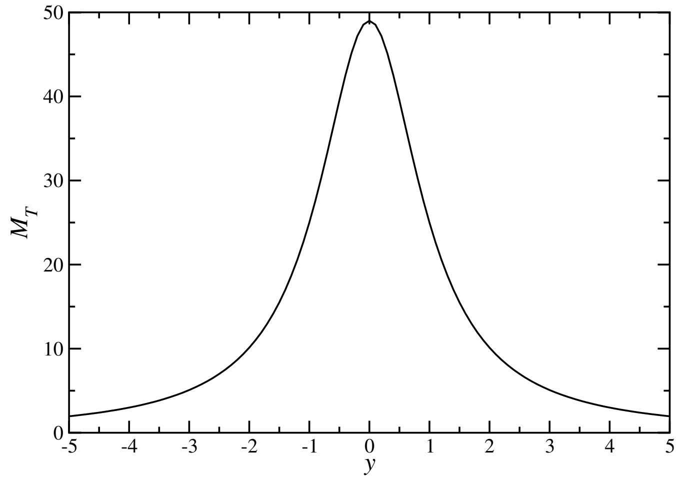

Figure 1. The reduced value of the total work M_T as a function of y for a fixed value of $x = 50$.

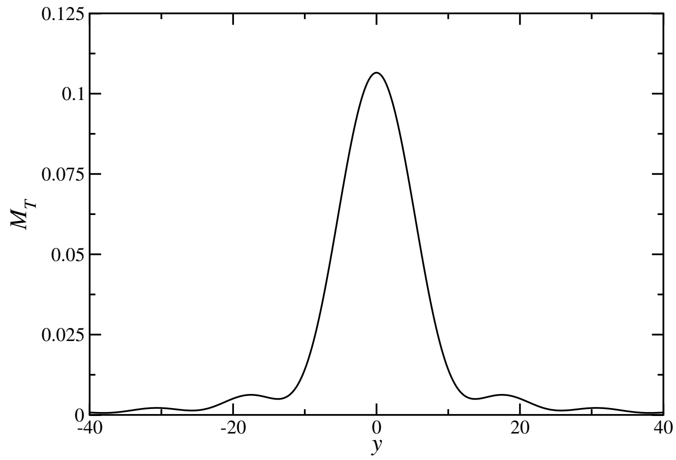

Figure 2. The reduced value of the total work M_T as a function of y for a fixed value of $x = 0.5$.

On the other hand, we can compare our result with that calculated in [14] in the case where the minimum in the potential is forced to obey circular motion and, in the absence of the magnetic field. In this case $\widetilde{\Omega} = 0$ and hence the frequency $\widehat{\Omega} = \Omega_0$, the effective friction coefficient $\gamma_e = \gamma$, the relaxation time $\widetilde{\tau}_r = \tau_r = \gamma/k$, and therefore equation (51) reduces to

$$M_T(\tau) = \psi \left\{ \tau - \tau_r \frac{2\tau_r \Omega_0 \, \mathrm{e}^{-\tau/\tau_r} \sin(\Omega_0 \tau)}{1 + \tau_r^2 \Omega_0^2} - \tau_r \frac{\left[1 - \tau_r^2 \Omega_0\right]\left[1 - \mathrm{e}^{-\tau/\tau_r} \cos(\Omega_0 \tau)\right]}{1 + \tau_r^2 \Omega_0^2} \right\}, \tag{54}$$

where

$$\psi = \frac{\beta \gamma \mathrm{r}^2 \Omega_0}{(1 + \tau_r^2 \Omega_0^2)}. \tag{55}$$

5.4. Oscillating electric field in the stationary case

In appendix B, we have calculated explicitly the work mean value and its corresponding variance which are given by

$$M_{s}(\tau) = 2\varpi_e\tau \tag{56}$$

$$V_{s}(\tau) = V_{T}(\tau) = 2M_{T}(\tau). \tag{57}$$

Now, the $\varepsilon(\tau)$ parameter is directly calculated as

$$\varepsilon(\tau) = \frac{\widetilde{\tau}_r}{\tau\left(1+\widetilde{\tau}_r^2\widehat{\Omega}^2\right)}\{2\widetilde{\tau}_r\widehat{\Omega}\,\mathrm{e}^{-\tau/\widetilde{\tau}_r}\sin(\widehat{\Omega}\tau) + \left[1-\widetilde{\tau}_r^2\widehat{\Omega}^2\right][1-\mathrm{e}^{-\tau/\widetilde{\tau}_r}\cos(\widehat{\Omega}\tau)]\}, \tag{58}$$

which also vanishes like $1/\tau$ as $\tau \to \infty$; therefore $V_S = 2M_{s}$ consistently with the SSFT. Also, this identity can be obtained from equation (57), in fact taking into account equation (51), it can be checked that $M_{T}(\tau) \to 2\varpi_e\tau = M_{s}(\tau)$ as $\tau \to \infty$.

5.5. The Jarzynski equality

It is well known that the JE [3] relates the nonequilibrium work done on a system W to the equilibrium free energy differences ΔF, that is $\langle \mathrm{e}^{-\beta W}\rangle = \mathrm{e}^{-\beta\Delta F}$. In the transient case, and according to equations (18) and (30), we can show that $\langle \mathrm{e}^{-\beta W_{\mathrm{tot}}}\rangle = 1$, from which we can conclude that $\Delta F = 0$, consistently with the BvL theorem in the absence of orbital diamagnetism in a classical system of charged particles in thermodynamic equilibrium [51].

6. Concluding remarks

We have verified the validity of the work-fluctuation theorems for a charged Brownian harmonic oscillator under the influence of an electromagnetic field. Though the results given in equations (31) and (45) may be seen as a generalization of the FTs to the case where an electromagnetic field is present, we have found an interesting effect caused by the magnetic field. First, we have shown that the work mean values given by equations (24) and (37) have a very similar algebraic structure than those given in [14], except by the time-varying function $\mathbf{U}^*(t) = \mathcal{R}(t)\mathbf{u}^*(t) = (q/k)\mathcal{R}(t)\dot{\mathbf{E}}(t)$, which is nothing else but the time-dependent rotation of the rate of change of the applied electric field. Similar situations occur for the variances given by equations (29) and (38). In the absence of the magnetic field ($C = 0$), the rotation matrix $\mathcal{R}(t)$ reduces to the unit matrix and therefore $\mathbf{U}^*(t) = \mathbf{u}^*(t)$, so that all of those results (24), (37), (29) and (38) are exactly the same as those obtained in [14], with arbitrary $\mathbf{u}^*(t)$. Our results have been obtained due to the fact that we have been able to show that the fluctuating variable $\mathbf{X}$ has a stationary state Gaussian distribution, which has been easily obtained from the explicit calculation of the Gaussian probability density for all time $t > 0$ given by equation (15). The initial distribution for the $\mathbf{X}_0$ variable is assumed to be Gaussian and corresponds to the same as that of the stationary state.

The results obtained in the two particular situations studied as applications in this work can be compared with those obtained in [14] in the absence of the magnetic field. Our theoretical results reduce exactly to those calculated in that reference, as it should be. On the other hand, equation (46) multiplied by the factor $k_B T$ is exactly the same work mean value calculated by Jayannavar and Sahoo [44] to verify the JE. In the stationary case, equations (46), (49), (56) and (57) establish that the SSFT theorem holds for times large enough, provided a stationary equilibrium state exists. It should be noted that the second application is referred to in terms of an oscillating electric field; this fact has allowed us to identify a kind of resonant behavior when both Larmor's frequency and the one corresponding to the electric field are equal. In this case, the work done on the system is symmetric with respect to the difference in frequencies and when they are equal, it presents a maximum. This effect, as well as the Hall fluctuation relations reported recently [11, 47], constitute some examples which are present because the magnetic field acts on the system.

Finally, as stated by van Zon and Cohen [14], the rectilinear motion model corresponds to the situation in the experiment of Wang *et al* [26], but they suggest that the circular motion might be implemented in a future experiment. In a similar way, our proposal might also be implemented in future experiments taking into account the presence of an electromagnetic field, in the two physical situations studied in this work, linear and oscillating electric field.

Acknowledgments

The authors wish to thank M Romero-Bastida for his help with the figures, A Aldama for proofreading the manuscript, and JL del Río for his comments on the fundamentals of equation (2)

Appendix A. Linear motion for the potential minimum

A.1. The transient case

For linear motion we have that $\mathbf{x}^* = (q/k)\bar{\mathbf{E}}(t) = (ut, ut)$ and thus $\mathbf{u}^* = u_{\rm opt}(1, 1)$, where $u_{\rm opt} = (q/k)\mathcal{E}$ and $\mathcal{E}$ is the amplitude of the electric field per unit time. In this case $\mathbf{U}^*(t) = \mathcal{R}(t)\mathbf{u}^*(t) = u_{\rm opt}\mathcal{R}(t)(1, 1)$. So the transitory mean value of the total work given by equation (24), after explicit calculations, reads

$$M_{\tau}(\tau) = 2\beta k u_{\rm opt}^2 \left\{ \int_0^{\tau} {\rm e}^{-\widetilde{\gamma}t'} \cos(\widetilde{\Omega}t')\,{\rm d}t' \int_0^{t'} {\rm e}^{\widetilde{\gamma}t''} \cos(\widetilde{\Omega}t'')\,{\rm d}t'' + \int_0^{\tau} {\rm e}^{-\widetilde{\gamma}t'} \sin(\widetilde{\Omega}t')\,{\rm d}t' \int_0^{t'} {\rm e}^{\widetilde{\gamma}t''} \sin(\widetilde{\Omega}t'')\,{\rm d}t'' \right\}, \tag{A.1}$$

where

$$\int_0^{t'} {\rm e}^{\widetilde{\gamma}t''} \cos(\widetilde{\Omega}t'')\,{\rm d}t'' = \frac{1}{2\widetilde{\gamma}_1}({\rm e}^{\widetilde{\gamma}_1 t'} - 1) + \frac{1}{2\widetilde{\gamma}_2}({\rm e}^{\widetilde{\gamma}_2 t'} - 1), \tag{A.2}$$

$$\int_0^{t'} {\rm e}^{\widetilde{\gamma}t''} \sin(\widetilde{\Omega}t'')\,{\rm d}t'' = \frac{-i}{2\widetilde{\gamma}_1}({\rm e}^{\widetilde{\gamma}_1 t'} - 1) + \frac{i}{2\widetilde{\gamma}_2}({\rm e}^{\widetilde{\gamma}_2 t'} - 1), \tag{A.3}$$

and $\widetilde{\gamma}_1 = \widetilde{\gamma} + {\rm i}\,\widetilde{\Omega}$ and $\widetilde{\gamma}_2 = \widetilde{\gamma} - {\rm i}\,\widetilde{\Omega}$. By substituting equations (A.2), (A.3) into equation (A.1) and after some algebra we have

$$M_{\tau}(\tau) = 2\beta k u_{\rm opt}^2 \left\{ \frac{1}{2}\left(\frac{1}{\widetilde{\gamma}_1} + \frac{1}{\widetilde{\gamma}_2}\right)\tau + \frac{1}{2\widetilde{\gamma}_1\widetilde{\gamma}_1}\left({\rm e}^{-\widetilde{\gamma}_1\tau} - 1\right) + \frac{1}{2\widetilde{\gamma}_2\widetilde{\gamma}_2}\left({\rm e}^{-\widetilde{\gamma}_2\tau} - 1\right) \right\}, \tag{A.4}$$

or explicitly

$$M_{\tau}(\tau) = 2\varpi\{\tau - \tau_r[1 - {\rm e}^{-\tau/\widetilde{\tau}_r}\cos(\widetilde{\Omega}\tau)] - 2\tau_r C\,{\rm e}^{-\tau/\widetilde{\tau}_r}\sin(\widetilde{\Omega}\tau) + \tau_r C^2[1 - {\rm e}^{-\tau/\widetilde{\tau}_r}\cos(\widetilde{\Omega}\tau)]\}, \tag{A.5}$$

where $\varpi = \beta\gamma u_{\rm opt}^2$. The variance, according to equation (30) of section 3, is then $V_{\tau}(\tau) = 2M_{\tau}(\tau)$.

A.2. The stationary case

In this case the mean value of the total work, according to equation (38), is given by

$$M_s(\tau) = 2\beta k u_{\mathrm{opt}}^2 \lim_{t\to\infty} \left\{ \int_t^{t+\tau} \mathrm{e}^{-\tilde{\gamma}t'} \cos(\tilde{\Omega}t')\,\mathrm{d}t' \int_0^{t'} \mathrm{e}^{\tilde{\gamma}t''} \cos(\tilde{\Omega}t'')\,\mathrm{d}t'' + \int_t^{t+\tau} \mathrm{e}^{-\tilde{\gamma}t'} \sin(\tilde{\Omega}t')\,\mathrm{d}t' \int_0^{t'} \mathrm{e}^{\tilde{\gamma}t''} \sin(\tilde{\Omega}t'')\,\mathrm{d}t'' \right\}. \tag{A.6}$$

In equation (A.6) we can use the results of equations (A.2) and (A.3), and after straightforward integrations we get

$$M_s(\tau) = 2\beta k u_{\mathrm{opt}}^2 \lim_{t\to\infty} \left\{ \frac{1}{2}\left(\frac{1}{\tilde{\gamma}_1} + \frac{1}{\tilde{\gamma}_2}\right)\tau + \frac{\mathrm{e}^{-\tilde{\gamma}_1 t}}{2\tilde{\gamma}_1\tilde{\gamma}_1}(\mathrm{e}^{-\tilde{\gamma}_1\tau} - 1) + \frac{\mathrm{e}^{-\tilde{\gamma}_1 t}}{2\tilde{\gamma}_2\tilde{\gamma}_2}(\mathrm{e}^{-\tilde{\gamma}_2\tau} - 1) \right\}, \tag{A.7}$$

and therefore as the time $t \to \infty$, it can easily be shown that $M_s(\tau) = 2\varpi\,\tau$. On the other hand, the variance in this stationary case must be calculated from equation (38) yielding the following:

$$V_s(\tau) = 4\beta k u_{\mathrm{opt}}^2 \lim_{t\to\infty} \left\{ \int_t^{t+\tau} \mathrm{e}^{-\tilde{\gamma}t'} \cos(\tilde{\Omega}t')\,\mathrm{d}t' \int_t^{t'} \mathrm{e}^{\tilde{\gamma}t''} \cos(\tilde{\Omega}t'')\,\mathrm{d}t'' + \int_t^{t+\tau} \mathrm{e}^{-\tilde{\gamma}t'} \sin(\tilde{\Omega}t')\,\mathrm{d}t' \int_t^{t'} \mathrm{e}^{\tilde{\gamma}t''} \sin(\tilde{\Omega}t'')\,\mathrm{d}t'' \right\}. \tag{A.8}$$

From this we show that

$$\int_t^{t'} \mathrm{e}^{\tilde{\gamma}t''} \cos(\tilde{\Omega}t'')\,\mathrm{d}t'' = \frac{1}{2\tilde{\gamma}_1}(\mathrm{e}^{\tilde{\gamma}_1 t'} - \mathrm{e}^{\tilde{\gamma}_1 t}) + \frac{1}{2\tilde{\gamma}_2}(\mathrm{e}^{\tilde{\gamma}_2 t'} - \mathrm{e}^{\tilde{\gamma}_2 t}), \tag{A.9}$$

$$\int_0^{t'} \mathrm{e}^{\tilde{\gamma}t''} \sin(\tilde{\Omega}t'')\,\mathrm{d}t'' = \frac{-\mathrm{i}}{2\tilde{\gamma}_1}(\mathrm{e}^{\tilde{\gamma}_1 t'} - \mathrm{e}^{\tilde{\gamma}_1 t}) + \frac{\mathrm{i}}{2\tilde{\gamma}_2}(\mathrm{e}^{\tilde{\gamma}_2 t'} - \mathrm{e}^{\tilde{\gamma}_1 t}). \tag{A.10}$$

By substituting equations (A.9) and (A.10) into equation (A.8), after evaluating the integrals and taking $t \to \infty$, it can be shown that

$$V_s(\tau) = 4\beta k u_{\mathrm{opt}}^2 \left\{ \frac{1}{2}\left(\frac{1}{\tilde{\gamma}_1} + \frac{1}{\tilde{\gamma}_2}\right)\tau + \frac{1}{2\tilde{\gamma}_1\tilde{\gamma}_1}(\mathrm{e}^{-\tilde{\gamma}_1\tau} - 1) + \frac{1}{2\tilde{\gamma}_2\tilde{\gamma}_2}(\mathrm{e}^{-\tilde{\gamma}_2\tau} - 1) \right\}. \tag{A.11}$$

If we compare equation (A.11) with equation (A.4), we conclude that $V_s(\tau) = 2M_{\scriptscriptstyle T}(\tau)$.

Appendix B. Circular motion for the potential minimum

B.1. The transient case

For the circular motion we also consider the time-dependent position vector $\mathbf{x}^*(t) = r(\sin\Omega_0 t, (1 - \cos\Omega_0 t))$ such that $r = qE/k$ is the radius of the circle and E the constant amplitude of the electric field. So the velocity is $\mathbf{u}^*(t) = r\Omega_0(\cos\Omega_0 t, \sin\Omega_0 t)$ and the total work mean value, according to equation (24), reads

$$M_{\scriptscriptstyle T}(\tau) = \beta k r^2 \Omega_0^2 \left\{ \int_0^{\tau} \mathrm{e}^{-\tilde{\gamma}t'} \cos(\hat{\Omega}t')\,\mathrm{d}t' \int_0^{t'} \mathrm{e}^{\tilde{\gamma}t''} \cos(\hat{\Omega}t'')\,\mathrm{d}t'' + \int_0^{\tau} \mathrm{e}^{\tilde{\gamma}t'} \sin(\hat{\Omega}t')\,\mathrm{d}t' \int_0^{t'} \mathrm{e}^{-\tilde{\gamma}t''} \sin(\hat{\Omega}t'')\,\mathrm{d}t'' \right\}, \tag{B.1}$$

where now $\widehat{\Omega} \equiv \Omega_0 - \widetilde{\Omega}$. We can see that the integrals of equation (B.1) have the same algebraic structure as those given in equation (A.1), and therefore we arrive at a similar expression as that displayed in equation (A.4). In this case the transient work mean value is then

$$M_T(\tau) = \frac{\beta k r^2 \Omega_0^2}{2}\left\{\left(\frac{1}{\widetilde{\Gamma}_1} + \frac{1}{\widetilde{\Gamma}_2}\right)\tau + \frac{1}{\widetilde{\Gamma}_1\widetilde{\Gamma}_1}(e^{-\widetilde{\Gamma}_1\tau} - 1) + \frac{1}{\widetilde{\Gamma}_2\widetilde{\Gamma}_2}(e^{-\widetilde{\Gamma}_2\tau} - 1)\right\}, \tag{B.2}$$

where $\widetilde{\Gamma}_1 = \widetilde{\gamma} + \mathrm{i}\,\widehat{\Omega}$ and $\widetilde{\Gamma}_2 = \widetilde{\gamma} - \mathrm{i}\,\widehat{\Omega}$. Explicitly this mean value can be written as

$$M_T(\tau) = \varpi_e\left\{\tau - \widetilde{\tau}_r\frac{2\widetilde{\tau}_r\widehat{\Omega}\,e^{-\tau/\widetilde{\tau}_r}\sin(\widehat{\Omega}\tau)}{1 + \widetilde{\tau}_r^2\widehat{\Omega}^2} - \widetilde{\tau}_r\frac{[1 - \widetilde{\tau}_r^2\widehat{\Omega}][1 - e^{-\tau/\widetilde{\tau}_r}\cos(\widehat{\Omega}\tau)]}{1 + \widetilde{\tau}_r^2\widehat{\Omega}}\right\}, \tag{B.3}$$

and ϖ_e is defined as

$$\varpi_e = \frac{\beta\gamma_e r^2\Omega_0^2}{\left(1 + \widetilde{\tau}_r^2\widehat{\Omega}^2\right)}. \tag{B.4}$$

The variance is again (see equation (30) $V_T(\tau) = 2M_T(\tau)$.

B.2. The stationary case

In this case we have from equation (37) that the work mean value is

$$M_S(\tau) = \beta k r^2\Omega_0^2 \lim_{t\to\infty}\left\{\int_t^{t+\tau} e^{-\widetilde{\gamma}t'}\cos(\widehat{\Omega}t')\,dt'\int_0^{t'} e^{\widetilde{\gamma}t''}\cos(\widehat{\Omega}t'')\,dt'' + \int_t^{t+\tau} e^{-\widetilde{\gamma}t'}\sin(\widehat{\Omega}t')\,dt'\int_0^{t'} e^{\widetilde{\gamma}t''}\sin(\widehat{\Omega}t'')\,dt''\right\}. \tag{B.5}$$

Again, the integrals of equation (B.5) are very similar to those given in equation (A.6), and therefore they also lead to a similar expression to that given by equation (A.7), that is

$$M_S(\tau) = \beta k r^2\Omega_0^2 \lim_{t\to\infty}\left\{\frac{1}{2}\left(\frac{1}{\widetilde{\Gamma}_1} + \frac{1}{\widetilde{\Gamma}_2}\right)\tau + \frac{e^{-\widetilde{\Gamma}_1 t}}{2\widetilde{\Gamma}_1\widetilde{\Gamma}_1}(e^{-\widetilde{\Gamma}_1\tau} - 1) + \frac{e^{-\widetilde{\Gamma}_1 t}}{2\widetilde{\Gamma}_2\widetilde{\Gamma}_2}(e^{-\widetilde{\Gamma}_2\tau} - 1)\right\}. \tag{B.6}$$

From this equation we easily check that as the time $t \to \infty$, $M_T(\tau) = \varpi_e\tau$.

Finally the variance, according to equation (38), is given by

$$V_S(\tau) = 2\beta k r^2\Omega_0^2 \lim_{t\to\infty}\left\{\int_t^{t+\tau} e^{-\widetilde{\gamma}t'}\cos(\widehat{\Omega}t')\,dt'\int_t^{t'} e^{\widetilde{\gamma}t''}\cos(\widehat{\Omega}t'')\,dt'' + \int_t^{t+\tau} e^{-\widetilde{\gamma}t'}\sin(\widehat{\Omega}t')\,dt'\int_t^{t'} e^{\widetilde{\gamma}t''}\sin(\widehat{\Omega}t'')\,dt''\right\}. \tag{B.7}$$

The integrals of equation (B.7) are quite similar to those given in equation (A.8), so that also after integrations and taking $t \to \infty$, we arrive at a similar expression to equation (A.11), that is

$$V_S(\tau) = \beta k r^2\Omega_0^2\left\{\left(\frac{1}{\widetilde{\Gamma}_1} + \frac{1}{\widetilde{\Gamma}_2}\right)\tau + \frac{1}{\widetilde{\Gamma}_1\widetilde{\Gamma}_1}(e^{-\widetilde{\Gamma}_1\tau} - 1) + \frac{1}{\widetilde{\Gamma}_2\widetilde{\Gamma}_2}(e^{-\widetilde{\Gamma}_2\tau} - 1)\right\}, \tag{B.8}$$

which according to equation (B.2) shows that $V_S(\tau) = 2M_T(\tau)$,

References

[1] Sevick E M, Prabhakar R, Williams S R and Searles D J 2008 *Annu. Rev. Phys. Chem.* **59** 603
[2] Joubaud S, Garnier N B and Ciliberto S 2007 *J. Stat. Mech.* P09018
[3] Jarzynski C 2007 *Lect. Notes Phys.* **711** 201
Jarzynski C 1997 *Phys. Rev. Lett.* **78** 2690
Jarzynski C 1997 *Phys. Rev.* E **56** 5018
[4] Horowitz J and Jarzynski C 2007 *J. Stat. Mech.* P11002
[5] Mai T and Dhar A 2007 *Phys. Rev.* E **75** 061101
[6] Touchette H and Cohen E G D 2007 *Phys. Rev.* E **76** 020101
[7] Baule A and Cohen E G D 2009 *Phys. Rev.* E **79** 030103
[8] Vilar J M G and Rubi J M 2008 *Phys. Rev. Lett.* **100** 020601
Vilar J M G and Rubi J M 2008 *Phys. Rev. Lett.* **101** 098902
Vilar J M G and Rubi J M 2008 *Phys. Rev. Lett.* **101** 098904
[9] Horowitz J and Jarzynski C 2008 *Phys. Rev. Lett.* **101** 098901
[10] Peliti L 2008 *Phys. Rev. Lett.* **101** 098903
[11] Roy D and Kumar N 2008 *Phys. Rev.* E **78** 052102
[12] Harris R and Schuetz G M 2007 *J. Stat. Mech.* P07020
[13] Bustamante C, Liphardt J and Ritort F 2005 *Phys. Today* **58** 43
[14] van Zon R and Cohen E G D 2003 *Phys. Rev.* E **67** 046102
[15] van Zon R and Cohen E G D 2004 *Phys. Rev.* E **69** 056121
[16] Seifert U 2005 *Phys. Rev. Lett.* **95** 040602
[17] Dhar A 2005 *Phys. Rev.* E **71** 036126
[18] Speck T and Seiferti U 2005 *Eur. Phys. J.* B **43** 529
[19] Mazonka O and Jarzynski C 1999 arXiv:cond-mat/9912121
[20] Evans D J and Searles D J 1994 *Phys. Rev.* E **50** 1645
[21] Crooks G E 1999 *Phys. Rev.* E **60** 2721
Crooks G E 2000 *Phys. Rev.* E **61** 2361
Crooks G E 1998 *J. Stat. Phys.* **90** 1481
[22] Gallavotti G and Cohen E G D 1995 *Phys. Rev. Lett.* **74** 2694
Gallavotti G and Cohen E G D 1995 *J. Stat. Phys.* **80** 931
[23] Lepri S, Rondoni L and Benetitin G 2000 *J. Stat. Phys.* **99** 857
[24] Searles D J and Evans D J 2000 *J. Chem. Phys.* **113** 3503
[25] Evans D J, Cohen E G D and Morriss G P 1993 *Phys. Rev. Lett.* **71** 2401
[26] Wang G M, Sevick E M, Mittag E, Searles D J and Evans D J 2002 *Phys. Rev. Lett.* **89** 050601
[27] Carberry D M, Reid J C, Wang G M, Sevick E M, Searles D J and Evans D J 2004 *Phys. Rev. Lett.* **92** 140601
[28] Trepagnier E H, Jarzynski C, Ritort F, Crooks G E, Bustamante C J and Liphardt J 2004 *Proc. Natl Acad. Sci.* **101** 15038
[29] Blickle V, Speck T, Helden L, Seifert U and Bechinger C 2006 *Phys. Rev. Lett.* **96** 070603
[30] Hou L J, Miskovic Z L, Piel A and Shukla P K 2009 *Phys. Plasmas* **16** 053705
[31] Morfilli G E and Ivlev A V 2009 *Rev. Mod. Phys.* **81** 1353
[32] Jiménez-Aquino J I, Velasco R M and Uribe F J 2008 *Phys. Rev.* E **77** 051105
[33] Holod I, Zagorodny A and Weiland J 2005 *Phys. Rev.* E **71** 046401
[34] Lev B I, Tymchyshyn V B and Zagorodny A G 2009 *Condens. Matter Phys.* **12** 593
[35] Consolini G, De Michelis P and Tozzi R 2008 *J. Geophys. Res.* **113** A08222
[36] Lechner W, Oberhofer H, Dellago C and Geissler P L 2006 *J. Chem. Phys.* **124** 044113
[37] Liphardt J, Dumont S, Smith S B, Tinoco I Jr and Bustamente C 2002 *Science* **296** 1832
[38] Narayan O and Dhar A 2004 *J. Phys. A: Math. Gen.* **37** 63
[39] Douarche F, Ciliberto S, Patrosyan A and Rabbiosi I 2005 *Europhys. Lett.* **70** 593
[40] Collin D, Ritort F, Jarzynski C, Smith S B, Tinoco I Jr and Bustamante C 2005 *Nature* **437** 231
[41] Tietz C, Schuler S, Speck T, Seifert U and Wrachtrup J 2006 *Phys. Rev. Lett.* **97** 050602
[42] Hatano T 1999 *Phys. Rev.* E **60** R5017
[43] Hatano T and Sasa S I 2001 *Phys. Rev. Lett.* **86** 3463
[44] Jayannavar A M and Sahoo M 2007 *Phys. Rev.* E **75** 032102
[45] Saha S and Jayannavar A M 2008 *Phys. Rev.* E **77** 022105
[46] Jiménez-Aquino J I, Velasco R M and Uribe F J 2008 *Phys. Rev.* E **78** 032102
[47] Jiménez-Aquino J I, Velasco R M and Uribe F J 2009 *Phys. Rev.* E **79** 061109
[48] Saito K and Dhar A 2007 *Phy. Rev. Lett.* **99** 180601

[49] Campisi M, Talkner P and Hänggi P 2009 *Phys. Rev. Lett.* **102** 210401
[50] Ritort F 2009 *Physics* **2** 43
[51] van Leeuwen J H 1921 *J. Phys.* **2** 3619
[52] Uhlenbeck G E and Ornstein L S 1930 *Phys. Rev.* **36** 823
[53] Chandrasekhar S 1943 *Rev. Mod. Phys.* **15** 1
[54] van Kampen N 2007 *Stochastic Processes in Physics and Chemistry* 3rd edn (Amsterdam: Elsevier)
[55] Risken H 1984 *The Fokker-Planck Equation: Methods of Solution and Applications* (Berlin: Springer)
[56] Taylor J B 1961 *Phys. Rev. Lett.* **6** 262
[57] Kurşunoğlu B 1962 *Ann. Phys.* **17** 259
[58] Karmeshu 1974 *Phys. Fluids* **17** 1828
[59] Czopnik R and Garbaczewski P 2001 *Phys. Rev.* E **63** 021105
[60] Simões T P and Lagos R E 2005 *Physica* A **355** 274
[61] Ford G W, Kac M and Mazur P 1965 *J. Math. Phys.* **6** 504
[62] Zwanzig R 1973 *J. Stat. Phys.* **9** 215
[63] Luczka J 2005 *Chaos* **15** 026107